Linear Equations and Their Graphs

3-1 Open Sentences in Two Variables

Objective To find solutions of open sentences in two variables and to solve problems involving open sentences in two variables.

Equations and inequalities such as

$$9x + 2y = 15, \quad y = x^2 - 4, \quad \text{and} \quad 2x - y \geq 6$$

are called **open sentences in two variables.**

A **solution** of an open sentence in the two variables x and y is a pair of numbers, one a value of x and the other a value of y, that together make the sentence a true statement. We usually write such a solution as an **ordered pair** in which the value of x is listed first and value of y is listed second. For example,

$$(1, 3) \quad \text{is a solution of} \quad 9x + 2y = 15 \quad \text{because} \quad 9(1) + 2(3) = 15.$$

However, $(3, 1)$ is *not* a solution because $9(3) + 2(1) \neq 15$. A solution of an open sentence is said to *satisfy* the sentence. The set of all solutions is called the **solution set** of the open sentence. Finding the solution set is called **solving** the open sentence.

Example 1 Solve the equation $9x + 2y = 15$ if the domain of x is $\{-1, 0, 1, 2, 3\}$.

Solution 1 Solve the equation for y.

$$y = \frac{15 - 9x}{2}$$

Then replace x with each value in its domain and find the corresponding value of y. The last column of the table lists the five solutions. Solving the equation over other domains will produce different solutions.

x	$\frac{15-9x}{2}$	y	Solution
-1	$\frac{15-9(-1)}{2}$	12	$(-1, 12)$
0	$\frac{15-9(0)}{2}$	$\frac{15}{2}$	$\left(0, \frac{15}{2}\right)$
1	$\frac{15-9(1)}{2}$	3	$(1, 3)$
2	$\frac{15-9(2)}{2}$	$-\frac{3}{2}$	$\left(2, -\frac{3}{2}\right)$
3	$\frac{15-9(3)}{2}$	-6	$(3, -6)$

$\therefore$ the solution set is $\left\{(-1, 12), \left(0, \frac{15}{2}\right), (1, 3), \left(2, -\frac{3}{2}\right), (3, -6)\right\}$. ***Answer***

Solution 2 Substitute each value in the domain of x in the given equation $9x + 2y = 15$. Then solve for y.

$$x = -1: \quad 9(-1) + 2y = 15 \qquad 2y = 24 \qquad y = 12$$

Solution $(-1, 12)$

$$x = 0: \quad 9(0) + 2y = 15 \qquad 2y = 15 \qquad y = \frac{15}{2}$$

Solution $\left(0, \frac{15}{2}\right)$

When you substitute the other values in the domain of x in the given equation, you will find the complete solution set:

$\left\{(-1, 12), \left(0, \frac{15}{2}\right), (1, 3), \left(2, -\frac{3}{2}\right), (3, -6)\right\}$. ***Answer***

Open sentences in two variables can be used to solve certain word problems.

Example 2 A customer asks a bank teller for \$390 in traveler's checks, some worth \$50 and some worth \$20. Find all possibilities for the number of each type of check the customer could receive.

Solution

Step 1 The problem asks for the number of \$50 checks and the number of \$20 checks whose total value is \$390.

Step 2 Let f = the number of \$50 checks and let t = the number of \$20 checks.

Step 3 The total value of the \$50 checks is $50f$ dollars and the total value of the \$20 checks is $20t$ dollars. Write an equation that expresses the total value of the checks in dollars.

$$50f + 20t = 390$$

Step 4 Solve the equation for one variable, say t, in terms of the other. First divide both sides by 10 to obtain this simpler equation:

$$5f + 2t = 39$$

$$t = \frac{39 - 5f}{2}$$

Remember: The number of each type of check *must* be a whole number. If f is even, then $5f$ is even and $39 - 5f$ is odd. If $39 - 5f$ is odd, t is not a whole number. Also, if f exceeds 7, t is negative. Therefore, replace f with 1, 3, 5, and 7 as shown in the table.

f	t	(f, t)
1	17	(1, 17)
3	12	(3, 12)
5	7	(5, 7)
7	2	(7, 2)

$\therefore$ the solution set of $50f + 20t = 390$ over the domain of the whole numbers is $\{(1, 17), (3, 12), (5, 7), (7, 2)\}$.

Step 5 The check is left for you.

$\therefore$ the customer can receive: 1 \$50 check and 17 \$20 checks
or 3 \$50 checks and 12 \$20 checks
or 5 \$50 checks and 7 \$20 checks
or 7 \$50 checks and 2 \$20 checks. ***Answer***

Example 3 Find all positive two-digit odd numbers with this property: When the digits are interchanged, the result exceeds the original number by more than 36.

Solution

Step 1 The problem asks for all positive two-digit odd numbers such that when the digits are interchanged, the new number is greater than the original number plus 36. Recall that every positive integer can be written in expanded form. For example:

$$73 = \underbrace{7}_{\text{tens' digit}} \cdot 10 + \underbrace{3}_{\text{units' digit}}$$

Step 2 Let u = the units' digit and let t = the tens' digit. Then the original number is $10t + u$. The number with the digits interchanged is $10u + t$.

Step 3 Write an inequality relating t and u.

new number	is greater than	original number	plus	36
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$10u + t$	$>$	$(10t + u)$	$+$	36

Step 4 Solve the inequality:

$$10u + t > 10t + u + 36$$
$$9u > 9t + 36$$

Divide both sides by 9:

$$u > t + 4$$

Since the original number has two digits, the tens' digit is not zero, so $t \neq 0$ and the replacement set for t is $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Since the number is odd, the replacement set for u is $\{1, 3, 5, 7, 9\}$.

t	$t + 4$	$u > t + 4$	(t, u)
1	5	7, 9	(1, 7), (1, 9)
2	6	7, 9	(2, 7), (2, 9)
3	7	9	(3, 9)
4	8	9	(4, 9)

If $t > 4$, then $u > 9$ and there are no more possibilities.

Step 5 The check is left for you.

$\therefore$ the numbers are 17, 19, 27, 29, 39, and 49. ***Answer***

Oral Exercises

Tell whether each ordered pair is a solution of the open sentence.

1. $3x + y = 1$ $(1, 0), (0, 1), (1, -2), (1, -4)$
2. $2x - 3y = 5$ $(-1, 1), (1, -1), (2, -3), (-5, -5)$
3. $7x - 2y = 8$ $(1, -1), (2, 3), (0, -4), (4, 10)$
4. $3x + 5y = 4$ $(-2, -2), (-2, 2), (8, 4), (7, -3)$
5. $5x - 2y > 6$ $(2, 0), (0, -3), (1, -1), (1, 1)$
6. $4x - 3y \le 2$ $(1, 2), (2, 1), (1, -1), (0, -1)$

Give three ordered pairs of integers that satisfy the open sentence.

7. $x - y = 2$ **8.** $5x + y = 6$ **9.** $2x + 3y > 10$ **10.** $3x + 5y \le 8$

11. Make up a word problem that could be solved using this equation: $10d + 25q = 160$.

Written Exercises

Solve each equation if the domain of x is $\{-1, 0, 2\}$.

A **1.** $2x + 3y = 7$ **2.** $3x + 6y = 9$ **3.** $-x - 2y = 0$

4. $-2x + y = -3$ **5.** $4x - 9y = 5$ **6.** $6x - \frac{1}{2}y = 3$

7–12. Solve each equation in Exercises 1–6 if the domain of x is $\{-2, 1, 3\}$.

Complete each ordered pair to form a solution of the equation.

Sample $x + 2y = 8$; $(0, \underline{\ ?\ })$, $(\underline{\ ?\ }, 0)$ **Solution** $(0, 4)$, $(8, 0)$

13. $3x + 2y = 12$ $(0, \underline{\ ?\ })$, $(\underline{\ ?\ }, 0)$, $(2, \underline{\ ?\ })$
14. $4x + 3y = 8$ $(0, \underline{\ ?\ })$, $(\underline{\ ?\ }, 0)$, $(5, \underline{\ ?\ })$
15. $5x - 2y = 7$ $(0, \underline{\ ?\ })$, $(\underline{\ ?\ }, 0)$, $(-3, \underline{\ ?\ })$
16. $x + 6y = -9$ $(0, \underline{\ ?\ })$, $(\underline{\ ?\ }, 0)$, $(-3, \underline{\ ?\ })$
17. $2x - 2y = 3$ $(1, \underline{\ ?\ })$, $\left(\frac{1}{2}, \underline{\ ?\ }\right)$, $\left(\underline{\ ?\ }, \frac{1}{2}\right)$
18. $3x + 5y = 3$ $(1, \underline{\ ?\ })$, $\left(-\frac{2}{3}, \underline{\ ?\ }\right)$, $\left(\underline{\ ?\ }, \frac{7}{5}\right)$
19. $\frac{1}{2}x - 2y = 1$ $(1, \underline{\ ?\ })$, $(6, \underline{\ ?\ })$, $(\underline{\ ?\ }, 0)$
20. $x + \frac{1}{3}y = 2$ $(1, \underline{\ ?\ })$, $(\underline{\ ?\ }, 6)$, $\left(\frac{1}{3}, \underline{\ ?\ }\right)$

Complete each map.

22) $4x - y = 2$

x	y
1	0
-1	2
1/2	-6

24) $x^2 - 1 = y$

x	y
-1	-1
0	0
1	3
2	

26) $3x + 2y = 15$

x	y
1	
3	
5	
7	

Solve each equation if each variable represents a *whole number*.

B **27.** $x + y = 4$

28. $2x + y = 6$

29. $4x + y = 15$

30. $x + 5y = 24$

31. $2x + 3y = 18$

32. $5x + 2y = 30$

Solve each open sentence if each variable represents a *positive integer*.

33. $x + y < 5$

34. $2x + y < 6$

35. $2x + 3y \leq 12$

36. $3x + 5y \leq 19$

37. $x + y^2 = 10$

38. $x^2 + 2y < 11$

In Exercises 39–40, the digits of a positive two-digit integer N are interchanged to form an integer K.

C **39.** Show that $N - K$ is an integral multiple of 9.

40. Show that $N + K$ is an integral multiple of 11.

Problems

In each problem (a) choose two variables to represent the numbers asked for, (b) write an open sentence relating the variables, and (c) solve the open sentence and give the answer to the problem. (Include solutions in which one of the variables is zero.)

A **1.** A bank teller needs to pay out \$75 using \$5 bills and \$20 bills. Find all possibilities for the number of each type of bill the teller could use.

2. Bruce, an appliance salesman, earns a commission of \$50 for each washing machine he sells and \$100 for each refrigerator. Last month he earned \$500 in commissions. Find all possibilities for the number of each kind of appliance he could have sold.

3. Luis has 95 cents in dimes and quarters. Find all possibilities for the number of each type of coin he could have.

4. Kimberly has \$1.95 in dimes and quarters. Find all possibilities for the number of each type of coin she could have.

5. A certain quadrilateral has three sides of equal length and its perimeter is 19 cm. Find all integral possibilities for the lengths of the sides in centimeters. (*Hint:* The sum of the lengths of any three sides of a quadrilateral must exceed the length of the fourth side.)

6. An isosceles triangle has perimeter 15m. Find all integral possibilities for the lengths of the sides in meters. (*Hint:* The sum of the lengths of any two sides of a triangle must exceed the third side.)

B 7. A box contains nickels, dimes, and quarters worth $2.00. Find all possibilities for the number of each coin if there are three more dimes than quarters.

8. A bag contains twice as many pennies as nickels and four more dimes than quarters. Find all possibilities for the number of each coin if their total value is $2.01.

In Exercises 9–12, the digits of a positive two-digit integer N are interchanged to form an integer K. Find all possibilities for N under the conditions described.

9. N is odd and exceeds K by more than 18.

10. The average of N, K, and 35 is 30.

C 11. The sum of K and twice N is less than 60.

12. N is even and exceeds K by more than 50.

Solve.

13. A 15-member special committee met three times. Twice as many members were present at the third meeting as at the first, and the average attendance was 9 people. Find all possibilities for the number of people present at each meeting.

14. A stick of wood is to be cut into three unequal pieces. The first piece is shorter than the second piece, and the second piece is shorter than the third. If the stick is 24 cm in length and the length of each piece is an even integer, what are the possibilities for the lengths of the pieces?

Mixed Review Exercises

Evaluate each expression if $x = -3$ and $y = 4$.

1. $2x - 5y$
2. $-x^2y$
3. $|x - y|$
4. $(x - 2)(y + 1)$
5. $\dfrac{3x + 1}{y}$
6. $|xy|$
7. $x + 3y$
8. $\dfrac{x - y}{x + y}$

Solve each open sentence and graph each solution set that is not empty.

9. $-7 < 2y - 5 \leq 3$
10. $|3 - m| > 1$
11. $3n + 7 \leq 8n - 13$